On Ratliff–Rush closure of modules

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§1 Introduction

Throughout my talk

- A a commutative Noetherian ring
- I, J ideals of A
- $\widetilde{I} = \bigcup_{\ell \geq 0} \left[I^{\ell+1} :_A I^{\ell} \right]$ the Ratliff–Rush closure of I
- $\mathcal{R}(I) = A[It] \subseteq A[t]$ the Rees algebra of I

Note that

- $I \subseteq \widetilde{I}$ and $\widetilde{I} \cdot \widetilde{J} \subseteq \widetilde{IJ}$
- $\widetilde{I} \subseteq \overline{I}$, if $\operatorname{grade}_A I > 0$
- If $J \subseteq I$ and J is a reduction of I, then $\widetilde{J} \subseteq \widetilde{I}$.

The projective scheme

$$\operatorname{Proj}\mathcal{R}(I) = \{P \in \operatorname{Spec}\mathcal{R}(I) \mid P \text{ is a graded ideal, } P \not\supseteq \mathcal{R}(I)_+\}$$

of $\mathcal{R}(I)$ defines the blowup of $\operatorname{Spec} A$ along $V(I)$.

Theorem 1.1 (Goto-Matsuoka, 2005)

Let $(A, \mathfrak{m})$ be a two-dimensional RLR, $\sqrt{I} = \mathfrak{m}$. Then TFAE.

(1) $\widetilde{I} = \overline{I}$.

(2) $\operatorname{Proj}\mathcal{R}(I)$ *is a normal scheme.*

When this is the case, $\mathcal{R}(I)$ has FLC, $\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(I)) \cong \mathcal{R}(\overline{I})/\mathcal{R}(I)$, and

$$\mathcal{R}(I) \text{ is CM} \iff \overline{I} = I.$$

- The notion of Rees algebra $\mathcal{R}(I)$ can be generalized to the module M, which is defined as

$$\mathcal{R}(M) = \mathrm{Sym}_A(M)/t(\mathrm{Sym}_A(M)).$$

- The Rees algebra of M includes the multi-Rees algebra, which corresponds to the case where $M = I_1 \oplus I_2 \oplus \cdots \oplus I_\ell$.
- The application to *equisingularity theory* needs this generalization ([2, 3]).

Question 1.2

Can we generalize Theorem 1.1 to the case of modules?

Contents

§2 Preliminaries

Setting 2.1

- A a Noetherian ring
- M a finitely generated A-module
- $F = A^{\oplus r}$ $(r > 0)$ s.t. $M \subseteq F$

Look at the diagram

$$\begin{tikzcd}
\mathrm{Sym}_A(M) \arrow[r, "\exists 1\, \mathrm{Sym}(i)"] & \mathrm{Sym}_A(F) = A[t_1, t_2, \ldots, t_r] =: S \\
M \arrow[u, "i"] \arrow[r, "i"] & F \arrow[u, "i"]
\end{tikzcd}$$

The Rees algebra $\mathcal{R}(M)$ of M is defined by

$$\begin{aligned}\mathcal{R}(M) &= \mathrm{Im}(\mathrm{Sym}(i)) \subseteq S = A[t_1, t_2, \dots, t_r] \\ &= \bigoplus_{n\geq 0} M^n.\end{aligned}$$

Definition 2.2

For $\forall n \geq 0$, we define

$$\overline{M^n} = \left(\overline{\mathcal{R}(M)}^S\right)_n \subseteq S_n = F^n$$

and call it *the integral closure of M^n*.

Proposition 2.3

For $\forall n \geq 0$, we have

$$\overline{M^n} = \left(\overline{(MS)^n}\right)_n .$$

In particular, $\overline{M} = \left(\overline{MS}\right)_1 \subseteq F$.

More precisely, $x \in \overline{M}$ satisfies

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0 \quad \text{in} \quad S$$

where $n > 0$, $c_i \in M^i$ for $1 \leq \forall i \leq n$.

Lemma 2.4

Suppose that $\operatorname{rank}_A M = r$. *Then* $\mathrm{Q}(\mathcal{R}(M)) = \mathrm{Q}(S)$. *Moreover, if* A *is a normal domain, then*

$$\overline{\mathcal{R}(M)}^{\mathrm{Q}(\mathcal{R}(M))} = \overline{\mathcal{R}(M)}^{S}$$

Proof.

Look at the diagram

$$\begin{tikzcd} \mathrm{Q}(A) \otimes_A \mathrm{Sym}_A(M) \arrow[r, "\cong"] & \mathrm{Q}(A) \otimes_A S \\ \mathrm{Sym}_A(M) \arrow[r, "\mathrm{Sym}(i)"] \arrow[u] & S \arrow[u] \end{tikzcd}$$

We get

$$0 \to t(\mathrm{Sym}_A(M)) \to \mathrm{Sym}_A(M) \to \mathcal{R}(M) \to 0$$

which yields

$$\mathrm{Q}(A) \otimes_A S \cong \mathrm{Q}(A) \otimes_A \mathrm{Sym}_A(M) \cong \mathrm{Q}(A) \otimes_A \mathcal{R}(M).$$

□

§3 Ratliff–Rush closure of modules

Setting 3.1

- A a Noetherian ring
- M a finitely generated A-module
- $F = A^{\oplus r}$ $(r > 0)$ s.t. $M \subseteq F$
- $\mathcal{R}(M) = \mathrm{Im}(\mathrm{Sym}_A(M) \longrightarrow \mathrm{Sym}_A(F)) \subseteq \mathrm{Sym}_A(F)$

We set $\mathfrak{a} = \mathcal{R}(M)_+ = \bigoplus_{n>0} M^n$, $S = \mathrm{Sym}_A(F)$, and

$$\widetilde{\mathcal{R}(M)}^S := \varepsilon^{-1}\left(\mathrm{H}^0_{\mathfrak{a}}(S/\mathcal{R}(M))\right) \subseteq S$$

where $\varepsilon : S \to S/\mathcal{R}(M)$.

Definition 3.2

For $\forall n \geq 0$, we define

$$\widetilde{M^n} = \left(\widetilde{\mathcal{R}(M)}^S\right)_n \subseteq S_n = F^n$$

and call it *the Ratliff–Rush closure of M^n*.

Definition 3.3 (Liu, 1998)

Suppose that A is a Noetherian domain. Then $\widetilde{M}$ is defined to be the largest A-submodule N of F satisfying

- $M \subseteq N \subseteq F$,
- $M^n = N^n$ for $\forall n \gg 0$.

Remark 3.4

These definitions coincide, when A is a Noetherian domain.

Proposition 3.5

For $\forall n \geq 0$, *we have*

$$\widetilde{M^n} = \bigcup_{\ell > 0} \left[(M^n)^{\ell+1} :_{F^n} (M^n)^{\ell} \right] = \left(\widetilde{(MS)^n} \right)_n .$$

In particular

$$\widetilde{M} = \bigcup_{\ell > 0} \left[M^{\ell+1} :_F M^{\ell} \right] = \left(\widetilde{MS} \right)_1 .$$

Corollary 3.6

Suppose that M *is a faithful* A*-module. Then*

$$\widetilde{M^n} \subseteq \overline{M^n} \subseteq F^n$$

for $\forall n \geq 0$. *Hence*

$$\mathcal{R}(M) \subseteq \widetilde{\mathcal{R}(M)}^S \subseteq \overline{\mathcal{R}(M)}^S \subseteq S.$$

Definition 3.7 (Buchsbaum-Rim, 1964, Hayasaka-Hyry, 2010)

Suppose that $(A, \mathfrak{m})$ is a Noetherian local ring with $d = \dim A$. Then M is called *a parameter module in F*, if

- $\ell_A(F/M) < \infty$,
- $M \subseteq \mathfrak{m}F$, and
- $\mu_A(M) = d + r - 1$.

Proposition 3.8

Suppose that $(A, \mathfrak{m})$ is a CM local ring with $d = \dim A > 0$. Let M be a parameter module in F. Then

$$\widetilde{M} = M.$$

Example 3.9

Let $A = k[[X, Y]]$. Set

$$M = \left\langle \begin{pmatrix} X \\ 0 \end{pmatrix}, \begin{pmatrix} Y \\ X \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix} \right\rangle \subseteq F = A \oplus A.$$

Then M is a parameter module in F and $\widetilde{M} = M$.

Example 3.10

Let $R = k[[X, Y, Z, W]]$. Set

$$A = R/(X, Y) \cap (Z, W), \quad Q = (X - Z, Y - W)A.$$

Then $\widetilde{Q} = Q$.

Proposition 3.11

Suppose that $L = Ax_1 + Ax_2 + \cdots + Ax_\ell \ (\subseteq M)$ *is a reduction of* M*. Then*

$$\widetilde{M} = \bigcup_{n>0} \left[M^{n+1} :_F ({Ax_1}^n + {Ax_2}^n + \cdots + {Ax_\ell}^n) \right].$$

Corollary 3.12

If L *is a reduction of* M*, then*

$$\widetilde{L} \subseteq \widetilde{M}.$$

Remark 3.13

The implication

$$L \subseteq M \implies \widetilde{L} \subseteq \widetilde{M}$$

does NOT hold in general.

Example 3.14 (Heinzer-Johnston-Lantz-Shah, 1993)

We consider

$$A = k[[t^3, t^4]] \subseteq k[[t]], \quad I = (t^8), \quad \text{and} \quad J = (t^{11}, t^{12}).$$

Then $J \subseteq I$, but $\widetilde{J} \not\subseteq \widetilde{I}$.

The following is the key in our argument.

Proposition 3.15

Suppose that M is a faithful A-module. Then the following assertions hold.

(1) $\widetilde{M^n} = (\widetilde{M})^n = M^n$ for $\forall n \gg 0$.

(2) Let N be an A-submodule of F s.t. $M \subseteq N$. Then TFAE.

  (i) $N \subseteq \widetilde{M}$.

  (ii) $M^\ell = N^\ell$ for $\exists\, \ell > 0$.

  (iii) $M^n = N^n$ for $\forall n \gg 0$.

  (iv) $\widetilde{M} = \widetilde{N}$.

(3) $\widetilde{\widetilde{M}} = \widetilde{M}$.

Let us note the following.

Lemma 3.16

Suppose that $(A, \mathfrak{m})$ is a Noetherian local ring. If $\overline{M} = F$, then $M = F$. In particular, if $M \neq F$ and M is faithful, then $\widetilde{M} \neq F$.

Proof.

Suppose $M \neq F$ and choose a counterexample M so that $r = \operatorname{rank}_A F > 0$ is as small as possible. Then $M \subseteq \mathfrak{m}F$. Therefore

$$F = \overline{M} \subseteq \overline{\mathfrak{m}F} = \overline{\mathfrak{m}}F = \mathfrak{m}F$$

so that $F = \mathfrak{m}F$, which is a contradiction. $\square$

In what follows, we assume

- $(A, \mathfrak{m})$ a Noetherian local ring with $d = \dim A$
- $F = A^{\oplus r}$ $(r > 0)$
- $(0) \neq M \subsetneq F$ s.t. $\ell_A(F/M) < \infty$

Then $\exists$ $\mathrm{br}_i(M) \in \mathbb{Z}$ $(0 \leq i \leq d + r - 1)$ s.t.

$$\ell_A(F^{n+1}/M^{n+1}) = \sum_{i=0}^{d+r-1} (-1)^i \cdot \mathrm{br}_i(M) \cdot \binom{n+d+r-i-1}{d+r-2}$$

for $\forall n \gg 0$.

The integer $\mathrm{br}_i(M)$ is called *the i-th Buchsbaum–Rim coefficient of M*.

Set

$$\mathcal{S} = \{N \subseteq F \mid M \subseteq N \subsetneq F,\ \mathrm{br}_i(M) = \mathrm{br}_i(N) \text{ for } 0 \leq \forall i \leq d+r-1\}.$$

Proposition 3.17

Suppose that M is a faithful A-module. Then

$$\widetilde{M} \in \mathcal{S} \quad \textit{and} \quad N \subseteq \widetilde{M} \quad \textit{for} \quad \forall N \in \mathcal{S}.$$

Hence $\widetilde{M}$ is the largest A-submodule N of F s.t.

- $M \subseteq N \subsetneq F$,
- $\mathrm{br}_i(M) = \mathrm{br}_i(N)$ for $0 \leq \forall i \leq d+r-1$.

§4 Main Results

Setting 4.1

- $(A, \mathfrak{m})$ a two-dimensional RLR, $|A/\mathfrak{m}| = \infty$
- $M \neq (0)$ a finitely generated torsion-free A-module
- $(-)^* = \mathrm{Hom}_A(-, A)$
- $F = M^{**} = A^{\oplus r}$ s.t. $\ell_A(F/M) < \infty$
- $\mathcal{R}(M)$ the Rees algebra of M
- $\mathfrak{M} = \mathfrak{m}\mathcal{R}(M) + \mathcal{R}(M)_+$
- $\mathrm{Proj}\,\mathcal{R}(M) = \{P \in \mathrm{Spec}\,\mathcal{R}(M) \mid P \text{ is a graded ideal, } P \not\supseteq \mathcal{R}(M)_+\}$

Note that $\dim \mathcal{R}(M) = r + 2$ and

$$\overline{M^n} = \left(\overline{M}\right)^n$$

for $\forall n \geq 0$.

The main result of my talk is stated as follows.

Theorem 4.2

TFAE.

(1) $\widetilde{M} = \overline{M}$.

(2) $\widetilde{M^n} = \overline{M^n}$ *for* $\forall n > 0$.

(3) $M^n = \overline{M^n}$ *for* $\exists n > 0$.

(4) $M^n = \overline{M^n}$ *for* $\forall n \gg 0$.

(5) $\operatorname{Proj} \mathcal{R}(M)$ *is a normal scheme.*

(6) $\mathcal{R}(M)_P$ *is normal for* $\forall P \in \operatorname{Spec} \mathcal{R}(M) \setminus \{\mathfrak{M}\}$.

When this is the case, $\mathcal{R}(M)$ *has FLC,* $\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M)) \cong \mathcal{R}(\overline{M})/\mathcal{R}(M)$*, and*

$$\mathcal{R}(M) \text{ is CM} \iff \overline{M} = M.$$

Proof of Theorem 4.2

(1) ⇒ (4) Note that $M^n = (\widetilde{M}\,)^n$ for $\forall n \gg 0$. Then

$$M^n = (\widetilde{M}\,)^n = (\overline{M})^n = \overline{M^n}.$$

(4) ⇒ (3) Obvious.

(3) ⇒ (1) Suppose $M^n = \overline{M^n} = (\overline{M})^n$ for $\exists n > 0$. Then $(\overline{M})^{n+1} = M^{n+1}$.
Therefore

$$\overline{M} \subseteq M^{n+1} :_F (\overline{M})^n = M^{n+1} :_F M^n \subseteq \widetilde{M} \subseteq \overline{M}$$

which yields $\widetilde{M} = \overline{M}$.

(1) ⇒ (2) We have $(\widetilde{M}\,)^n = (\overline{M})^n$ for $\forall n > 0$. Then

$$\overline{M^n} = (\overline{M})^n = (\widetilde{M}\,)^n \subseteq \widetilde{M^n} \subseteq \overline{M^n}$$

as desired.

(2) ⇒ (1) Obvious.

Theorem 4.2

TFAE.

(1) $\widetilde{M} = \overline{M}$.

(2) $\widetilde{M^n} = \overline{M^n}$ *for* $\forall n > 0$.

(3) $M^n = \overline{M^n}$ *for* $\exists n > 0$.

(4) $M^n = \overline{M^n}$ *for* $\forall n \gg 0$.

(5) $\operatorname{Proj} \mathcal{R}(M)$ *is a normal scheme.*

(6) $\mathcal{R}(M)_P$ *is normal for* $\forall P \in \operatorname{Spec} \mathcal{R}(M) \setminus \{\mathfrak{M}\}$.

When this is the case, $\mathcal{R}(M)$ *has FLC,* $\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M)) \cong \mathcal{R}(\overline{M})/\mathcal{R}(M)$, *and*

$$\mathcal{R}(M) \text{ is CM} \quad \Longleftrightarrow \quad \overline{M} = M.$$

(4) $\Rightarrow$ (6) Suppose $M^n = \overline{M^n}$ for $\forall n \gg 0$. Let $C = \mathcal{R}(\overline{M})/\mathcal{R}(M)$. Then $C_n = (0)$ for $n \gg 0$, so that C is finitely graded. Therefore

$$\mathfrak{a}^m \cdot C = (0), \quad \mathfrak{m}^m \cdot C = (0)$$

for $\exists\ m > 0$. Thus $\mathfrak{M} \subseteq \sqrt{(0) : C}$ and hence

$$\mathrm{Supp}_{\mathcal{R}(M)} C \subseteq \{\mathfrak{M}\}.$$

Consequently, for $\forall P \in \operatorname{Spec} \mathcal{R}(M) \setminus \{\mathfrak{M}\}$, $\mathcal{R}(M)_P = \mathcal{R}(\overline{M})_P$ is normal.

(6) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (4) Let $C = \mathcal{R}(\overline{M})/\mathcal{R}(M)$. We can check that

$$\mathfrak{a} \subseteq \sqrt{(0) : C}$$

whence C is finitely graded. Hence $M^n = \overline{M^n}$ for $\forall n \gg 0$.

Choose a parameter module L in F s.t. L is a reduction of $\overline{M}$. Then

$$(\overline{M})^2 = L \cdot \overline{M}$$

so that $\mathcal{R}(\overline{M})$ is a CM ring. Therefore

$$\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M)) \cong \mathcal{R}(\overline{M})/\mathcal{R}(M), \quad \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(M)) = (0) \quad \text{for} \quad \forall i \neq 1, r+2.$$

Hence $\mathcal{R}(M)$ has FLC and

$$\begin{aligned} \mathcal{R}(M) \text{ is a CM ring} \quad &\Longleftrightarrow \quad \mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M)) = (0) \\ &\Longleftrightarrow \quad (\overline{M})^n = M^n \text{ for } \forall n > 0 \\ &\Longleftrightarrow \quad \overline{M} = M \end{aligned}$$

which complete the proof. $\square$

Corollary 4.3

Suppose that $M \neq F$ *and* $\widetilde{M} = \overline{M}$*. Then*

$$\mathrm{br}_1(M) = \mathrm{br}_0(M) - \ell_A(F/\overline{M}), \quad \mathrm{br}_i(M) = 0 \ \ \textit{for} \ \ 2 \leq \forall i \leq r+1$$

and

$$\ell_A(F^{n+1}/(\overline{M})^{n+1}) = \mathrm{br}_0(M) \cdot \binom{n+r+1}{r+1} - \mathrm{br}_1(M) \cdot \binom{n+r}{r} \ \ \textit{for} \ \ \forall n \geq 0.$$

§5 Application

We maintain the notation as in Setting 4.1.

Theorem 5.1

TFAE.

(1) $\mathcal{R}(M)$ *is a Buchsbaum ring and* $\widetilde{M} = \overline{M}$.

(2) $\mathcal{R}(M)$ *is a Buchsbaum ring and* $\operatorname{Proj} \mathcal{R}(M)$ *is normal.*

(3) $\mathfrak{m}\overline{M} \subseteq M$ *and* $M \cdot \overline{M} = M^2$.

When this is the case,

$$\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M)) = \left[\mathrm{H}^1_{\mathfrak{M}}(\mathcal{R}(M))\right]_1 \cong \overline{M}/M$$

and $\overline{M^n} = M^n$ *for* $\forall n \geq 2$.

Example 5.2

Let $A = k[[X, Y]]$. Set

$$I = (X^4, X^3Y^2, XY^6, Y^8) \quad \text{and} \quad M = I \oplus I \subseteq F = A \oplus A.$$

Then $\widetilde{M} = \overline{M}$, but $\mathcal{R}(M)$ is not Buchsbaum.

Example 5.3

Let $A = k[[X, Y]]$. Set

$$I_1 = (X^6, X^5Y^2, X^4Y^3, X^3Y^4, XY^7, Y^8), \quad I_2 = (X^5, X^4Y^2, X^3Y^3, XY^6, Y^7)$$

and

$$M = I_1 \oplus I_2 \subseteq F = A \oplus A.$$

Then $\widetilde{M} = \overline{M}$ and $\mathcal{R}(M)$ is a Buchsbaum ring.

Corollary 5.4

Suppose that $\mathcal{R}(M)$ is a Buchsbaum ring and $\widetilde{M} = \overline{M}$. Then, for $\forall I \subsetneq A$ an ideal of A s.t. $\sqrt{I} = \mathfrak{m}$ and $\overline{I} = I$,

$$\mathcal{R}(I \cdot M) \text{ is a Buchsbaum ring.}$$

In particular, $\mathcal{R}(\mathfrak{m}^\ell M)$ is a Buchsbaum ring for $\forall \ell \geq 0$.

Corollary 5.5

Let $M_1, M_2 \neq (0)$ be finitely generated torsion-free A-modules. We set

$$F_1 = (M_1)^{**}, \quad F_2 = (M_2)^{**}$$

and

$$M = M_1 \oplus M_2 \subseteq F = F_1 \oplus F_2.$$

Then TFAE.

(1) $\mathcal{R}(M)$ *is a Buchsbaum ring and* $\widetilde{M} = \overline{M}$.

(2) $\mathcal{R}(M_i)$ *is a Buchsbaum ring,* $\widetilde{M_i} = \overline{M_i}$ $(i = 1, 2)$, *and*

$$M_1 \cdot \overline{M_2} = \overline{M_1} \cdot M_2 = M_1 \cdot M_2.$$

Corollary 5.6

Suppose that $\mathcal{R}(M)$ is a Buchsbaum ring and $\widetilde{M} = \overline{M}$. Then

$$\mathcal{R}(N) \text{ is a Buchsbaum ring } \text{ and } \widetilde{N} = \overline{N}.$$

for all direct summand N of M.

Corollary 5.7

Suppose that $\mathcal{R}(M)$ is a Buchsbaum ring and $\widetilde{M} = \overline{M}$. Then

$$\mathcal{R}(M^{\oplus \ell}) \text{ is a Buchsbaum ring } \text{ and } \widetilde{M^{\oplus \ell}} = \overline{M^{\oplus \ell}}$$

for $\forall \ell > 0$.

Example 5.8

Let $A = k[[X, Y]]$ and k an infinite field. Set

$$M = \left\langle \begin{pmatrix} X^3 \\ 0 \end{pmatrix}, \begin{pmatrix} X^2Y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} XY^3 \\ X^3 \end{pmatrix}, \begin{pmatrix} Y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X^2Y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ XY^4 \end{pmatrix}, \begin{pmatrix} 0 \\ Y^5 \end{pmatrix} \right\rangle.$$

Then $\widetilde{M} = \overline{M}$, $\mathcal{R}(M)$ is a Buchsbaum ring, and M is indecomposable.

We set

$$\mathcal{F}(M) = A/\mathfrak{m} \otimes_A \mathcal{R}(M) \cong \mathcal{R}(M)/\mathfrak{m}\mathcal{R}(M)$$

and call it *the fiber cone of M*.

Note that

$$\dim \mathcal{F}(M) = r + 1.$$

Theorem 5.9

Suppose that $\mathcal{R}(M)$ is a Buchsbaum ring and $\widetilde{M} = \overline{M}$. Then

$$\mathcal{F}(M) \text{ is a Buchsbaum ring.}$$

Thank you so much for your attention.

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